ON THE MOTION OF A GYROCOMPASS WITH RANDOM DISPLACEMENTS OF ITS POINT OF SUPPORT

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The paper investigates the motion of a gyrocompass with random displacements of its point of support, which takes place in the rolling of ships, the vibration of the instrument base and in other cases. Under these conditions, the acceleration of the point of support of the gyrocompass represents a random vector function of time and the motion of the gyrocompass will be described by a system of linear nonhomogeneous differential equations with random coefficients and random right-hand sides, i.e. under these conditions the gyrocompass represents a system with random parametric excitation under the action of random external forces.

1. In the derivation of the equations of motion of the gyrocompass for the case under consideration, we shall proceed form the general equations of motion of a gyrocompass, obtained by Ishlinskii [1], which can be written in the following form:

$$\begin{gathered}
2B\cos(\varepsilon-\delta)\,[-(V/R)(\sin\alpha_1\sin\gamma-\cos\alpha_1\sin\beta\cos\gamma)-(\alpha_1^{\cdot}+\Omega)\cos\beta\cos\gamma- \\
-\beta^{\cdot}\sin\gamma]=\mathrm{lm}\,W_1^{\circ}\sin\alpha_1\cos\beta-\mathrm{lm}\,W_2^{\circ}\cos\alpha_1\cos\beta-l(mW_3^{\circ}+P)\sin\beta-M_x^{*} \\
2B\cos(\varepsilon-\delta)\,[(V/R)(\sin\alpha_1\cos\gamma+\cos\alpha_1\sin\beta\sin\gamma)-(\alpha^{\cdot}+\Omega)\cos\beta\sin\gamma+ \\
+\beta^{\cdot}\cos\gamma]=M_y^{*} \qquad (1.1) \\
[2B\cos(\varepsilon-\delta)]^{\cdot}=\mathrm{lm}\,W_1^{\circ}(\cos\alpha_1\cos\gamma-\sin\alpha_1\sin\beta\sin\gamma)+ \\
+\mathrm{lm}\,W_2^{\circ}(\sin\alpha_1\cos\gamma+\cos\alpha_1\sin\beta\sin\gamma)-l(mW_3^{\circ}+P)\cos\beta\sin\gamma+M_z^{*} \\
2B\sin(\varepsilon-\delta)\,[(V/R)\cos\alpha_1\cos\beta+(\alpha_1^{\cdot}+\Omega)\sin\beta+\gamma^{\cdot}]=\varkappa\sin\delta\cos\delta-M_{y_1}^{*}
\end{gathered}$$

Above, a dot represents differentiation with respect to time.

Equations (1.1) describe the motion of the gyrocompass relative to a system of coordinates $\xi_0\eta_0\zeta$, oriented so, that one of the horizontal coordinates (ξ_0-axis) is directed along the horizontal component V of the absolute ship velocity. The ζ-axis is directed upwards along the radius of the Earth globe. The coordinates of the axes ξ_0 and η_0 are turned with respect to the geographic axes ξ and η (oriented, respectively, towards east and north) by an angle σ, found from

$$\sin\sigma=\frac{v_N}{V},\quad \cos\sigma=\frac{RU\cos\varphi+v_E}{V},\quad V=\sqrt{(RU\cos\varphi+v_E)^2+v_N^{\,2}} \qquad (1.2)$$

Here, R is the distance of the point of support of the gyrocompass from the center of the Earth, U is the angular velocity of the daily rotation of the Earth, φ is the latitude of the point of observation and v_E and v_N are the eastern and northern components of the relative ship velocity.

Let α_1, β and γ denote Euler's angles, which determine the position of the gyrosphere relative to the coordinate trihedron $\xi_0\eta_0\zeta$, α_1 denote the angle of rotation of the gyrosphere about axis ζ, β the angle of rotation about the negative direction of x^*-axis (nodal line) and γ the angle of rotation about the z-axis rigidly connected to the gyrosphere. Angle α, the rotation of the gyrocompass in azimuth, considered relative to the northern direction, will thus be

$$\alpha = \alpha_1 + \sigma \tag{1.3}$$

The z-axis is directed along the bisectrix of the angle, formed by the axes of the gyroscope rotors, while x and y denote axes rigidly connected to the gyrosphere, x being in the equatorial plane of the gyrosphere (i.e. in the plane formed by the axes of the gyroscope rotors) and y directed upwards along the normal to the equatorial plane.

Let $\varepsilon - \delta$ denote the angle between the axis of the rotor of one of the gyroscopes in the gyrosphere and the z-axis of the gyrosphere. The rotor axis of the second gyroscope is symmetrically placed. The axes of the gyroscope casings are interconnected by an antiparallelogram. In the state of equilibrium, when the gyroscope rotors do not gyrate, the angle between the rotor axes is equal to 2ε, therefore δ represents the angle of precession of the gyroscopes relative to the gyrosphere.

Let W_1°, W_2° and W_3° be the components of acceleration of the gyrocompass point of support along axes ξ_0, η_0 and ζ, respectively. In their determination it must be remembered that in the problem under consideration $v_\zeta = R^{\cdot} \neq 0$.

The projection of the instantaneous angular velocity of the coordinate trihedron $\xi_0\eta_0\zeta$ on the ζ-axis, is denoted by Ω, where

$$\Omega = U \sin\varphi + \frac{v_E}{R} \tan\varphi + \sigma^{\cdot} \tag{1.4}$$

Let B denote the individual kinetic moment of each of the two gyroscopes installed in the gyrosphere. The coefficient $\varkappa$ in the fourth equation of (1.1), determines the stiffness of the springs connecting the gyroscope casings with the gyrosphere. The quantities m and P represent, respectively, the mass of the gyrosphere with all its internal elements and the attraction force acting on it, while l is the distance between the center of gravity of the gyrosphere, situated on y-axis, and its geometric center, i.e. the point of support of the gyrocompass.

The moments of all the other external forces, not taken into account in Equations (1.1), about the axes x, y and z, rigidly connected to the gyrosphere, are denoted, respectively, by $M_x{}^*$, $M_y{}^*$ and $M_z{}^*$, while $M_{y_1}{}^*$ represents the moment of these forces relative to the axis of the gyroscope casing, about which the angle of precession δ is read.

We shall now modify the system if equations (1.1). The first equation of (1.1) is multiplied by $\cos\gamma$, the second equation by $\sin\gamma$ and the two new equations are added. Further, the first equation of (1.1) is multiplied by $-\sin\gamma$, the second by $\cos\gamma$ and the equations added again. Substituting the first two equations of (1.1) by the above two new equations, we get the equivalent system of equations

$$\begin{aligned}
&2B\cos(\varepsilon-\delta)\,[(V/R)\cos\alpha_1\sin\beta-(\alpha_1^{\cdot}+\Omega)\cos\beta] = lmW_1^\circ\sin\alpha_1\cos\beta\cos\gamma - \\
&\qquad - lmW_2^\circ\cos\alpha_1\cos\beta\cos\gamma - l\,(mW_3^\circ+P)\sin\beta\cos\gamma - M_x^*\\
&2B\cos(\varepsilon-\delta)\,((V/R)\sin\alpha_1+\beta^{\cdot}) = -lmW_1^\circ\sin\alpha_1\cos\beta\sin\gamma + \\
&\qquad + lmW_2^\circ\cos\alpha_1\cos\beta\sin\gamma + l\,(mW_3^\circ+P)\sin\beta\sin\gamma + M_y^*\\
&[2B\cos(\varepsilon-\delta)]^{\cdot} = lmW_1^\circ(\cos\alpha_1\cos\gamma-\sin\alpha_1\sin\beta\sin\gamma) + lmW_2^\circ(\sin\alpha_1\cos\gamma + \\
&\qquad + \cos\alpha_1\sin\beta\sin\gamma) - l\,(mW_3^\circ+P)\cos\beta\sin\gamma + M_z^*\\
&2B\sin(\varepsilon-\delta)\,[(V/R)\cos\alpha_1\cos\beta+(\alpha_1^{\cdot}+\Omega)\sin\beta+\gamma^{\cdot}] = \varkappa\sin\delta\cos\delta - M_{y_1}^*
\end{aligned} \tag{1.5}$$

where

$$M^*_{x^*} = M_x{}^* \cos \gamma - M_y{}^* \sin \gamma, \qquad M^*_{y^*} = M_x{}^* \sin \gamma + M_y{}^* \cos \gamma \tag{1.6}$$

Since the axes x and y are turned relative to x^* and y^* by an angle γ, measured counterclockwise from x^*-axis it follows from (1.6) that $M^*_{x^*}$ and $M^*_{y^*}$ represent the moments of the above mentioned external forces about axes x^* and y^*.

Considering now Equations (1.5) and noting that $RU \cos \varphi$ is very much larger than v_E and v_N, we shall substitute (1.2) and (1.4) with the following approximate relations:

$$V \approx RU \cos \varphi, \qquad \sigma \approx 0, \quad \varphi \approx \text{const}, \qquad \Omega \approx U \sin \varphi \tag{1.7}$$

When the base is at rest with respect to the Earth, the acceleration components, of the gyrocompass point of support, along axes ξ, η and ζ are

$$W_1{}^\circ = 0, \quad W_2{}^\circ = V\Omega = RU^2 \sin \varphi \cos \varphi, \quad W_3{}^\circ = -\frac{V^2}{R} = -RU^2 \cos^2 \varphi \tag{1.8}$$

It can therefore be assumed, that with random displacements of the gyrocompass point of support

$$W_1{}^\circ = W_1, \qquad W_2{}^\circ = RU^2 \sin \varphi \cos \varphi + W_2, \quad W_3{}^\circ = -RU^2 \cos^2 \varphi + W_3 \tag{1.9}$$

Here, W_1, W_2 and W_3 are random processes with zero mathematical expectations.

In normal gyrocompasses, the spring stiffness $\varkappa$ is several times larger than in horizontal gyrocompasses. Therefore in gyrocompasses (even with $v_\zeta \equiv 0$) ballistic deviations take place. To minimize them, the parameters of the compass are chosen to satisfy the condition

$$2B \cos (\varepsilon - \delta^*) = \operatorname{lm} RU \cos \varphi \tag{1.10}$$

where δ^* is the value of the angle δ for the case of the stationary motion of the gyrocompass, installed on a base fixed with respect to the Earth.

When condition (1.10) is satisfied, according to (1.4), (1.5) and (1.8), the position of equilibrium of a gyrocompass on a fixed base with respect to the Earth, for the case when $M_x{}^* = M_y{}^* = M_z{}^* = M_{y_1}{}^* = 0$, will be given by

$$\alpha_1 = \beta = \gamma = 0, \qquad \delta = \delta^* \tag{1.11}$$

where δ^* satisfies the following relation, obtained from the fourth equation of (1.5)

$$2B \sin (\varepsilon - \delta^*) \, U \cos \varphi = \varkappa \sin \delta^* \cos \delta^* \tag{1.12}$$

Now, denoting

$$\delta_1 = \delta - \delta^* \tag{1.13}$$

and restricting the study to small vibrations of the system, about the equilibrium position, we shall consider angles α_1, β, γ and δ_1 to be small. In this case, keeping only the terms of the first oeder of magnitude, we obtain the following relations:

$$\begin{aligned} &2B \cos (\varepsilon - \delta) = \Xi_1 + \Xi_2 \delta_1, \qquad [2B \cos (\varepsilon - \delta)]^\cdot = \Xi_2 \delta_1{}^\cdot \\ &2B \sin (\varepsilon - \delta) = \Xi_2 - \Xi_1 \delta_1, \\ &\varkappa \sin \delta \cos \delta = \varkappa \sin \delta^* \cos \delta^* + (\varkappa \cos 2\delta^*) \, \delta_1 \end{aligned} \tag{1.14}$$

where

$$\Xi_1 = 2B \cos (\varepsilon - \delta^*), \quad \Xi_2 = 2B \sin (\varepsilon - \delta^*) \tag{1.15}$$

Assuming, further, that

$$P - mRU^2 \cos^2 \varphi \approx P \tag{1.16}$$

denoting

$$\nu^2 = \frac{g}{R}, \qquad \mu^2 = \frac{lP \, (\operatorname{lm} RU^2 \cos^2 \varphi + \varkappa \cos 2\delta^*)}{\Xi_2{}^2} \tag{1.17}$$

(note [2], that in normal gyrocompasses $\mu \approx 5\nu$) and, introducing new variables

$$x_1 = U \cos \varphi\, \alpha_1, \quad x_2 = \beta, \quad x_3 = \gamma, \quad x_4 = (\Xi_2/\Xi_1)\, U \cos \varphi \delta_1 \tag{1.18}$$

we obtain according to (1.5) the following differential equations, describing the motion of the gyrocompass with random displacements of its point of support:

$$\frac{dx_1}{dt} + \frac{1}{RU \cos \varphi} W_1 x_1 - \left(\nu^2 + \frac{1}{R} W_3\right) x_2 + \Omega x_4 = \frac{1}{R} W_2 + y_1(t)$$

$$\frac{dx_2}{dt} + x_1 - \left(\Omega + \frac{1}{RU \cos \varphi} W_2\right) x_3 = y_2(t), \qquad \frac{dx_3}{dt} + \Omega x_2 - \frac{\mu^2}{\nu^2} x_4 = y_3(t)$$

$$\frac{dx_4}{dt} - \left(\Omega + \frac{1}{RU \cos \varphi} W_2\right) x_1 + \left(\nu^2 + \frac{1}{R} W_3\right) x_3 = \frac{1}{R} W_1 + y_4(t) \tag{1.19}$$

where

$$y_1(t) = \frac{1}{\operatorname{Im} R} M_{x_*}^*, \qquad y_2(t) = \frac{1}{\operatorname{Im} RU \cos \varphi} M_{y_*}^*$$
$$y_3(t) = -\frac{1}{\Xi_2} M_{y_1}^*, \qquad y_4 = \frac{1}{\operatorname{Im} R} M_z^* \tag{1.20}$$

The system (1.19) represents a system of linear differential equations with random coefficients and random right-hand sides.

2. In the case when, in addition to the forces clearly indicated in Equations (1.1), there are no additional external forces applied to the gyroscope we must assume in Equations (1.1)

$$M_x^* = M_y^* = M_z^* = M_{y_1}^* = 0 \tag{2.1}$$

It follows that

$$y_i(t) \equiv 0 \qquad (i = 1, \ldots, 4)$$

Assuming that conditions (2.1) are satisfied, and denoting by λ the small parameter contained in Equations (1.19) $\lambda = R_0/R$, where R_0 is some normalizing coefficient, satisfying, for example, the condition

$$|W_i|_{\max} / R_0 = 1 \text{ sec}^{-2},$$

we can express Equations (1.19) in the following form:

$$\frac{dx_1}{dt} - \nu^2 x_2 + \Omega x_4 + \lambda \left(\frac{1}{R_0 U \cos \varphi} W_1 x_1 - \frac{1}{R_0} W_3 x_2\right) = \lambda \frac{1}{R_0} W_2 \tag{2.2}$$

$$\frac{dx_2}{dt} + x_1 - \Omega x_3 - \lambda \frac{1}{R_0 U \cos \varphi} W_2 x_3 = 0, \qquad \frac{dx_3}{dt} + \Omega x_2 - \frac{\mu^2}{\nu^2} x_4 = 0$$

$$\frac{dx_4}{dt} - \Omega x_1 + \nu^2 x_3 - \lambda \left(\frac{1}{R_0 U \cos \varphi} W_2 x_1 - \frac{1}{R_0} W_3 x_3\right) = \lambda \frac{1}{R_0} W_1$$

We shall write the system (2.2) in matrix form

$$\frac{dx}{dt} + ax = -\lambda y(t)\, x + \lambda z(t) \tag{2.3}$$

Here

$$x = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix}, \qquad a = \begin{Vmatrix} 0 & -\nu^2 & 0 & \Omega \\ 1 & 0 & -\Omega & 0 \\ 0 & \Omega & 0 & -\dfrac{\mu^2}{\nu^2} \\ -\Omega & 0 & \nu^2 & 0 \end{Vmatrix} \tag{2.4}$$

$$y(t) = \begin{Vmatrix} s_1 W_1 & -s_2 W_3 & 0 & 0 \\ 0 & 0 & -s_1 W_2 & 0 \\ 0 & 0 & 0 & 0 \\ -s_1 W_2 & 0 & s_2 W_3 & 0 \end{Vmatrix}, \qquad z(t) = \begin{Vmatrix} s_2 W_2 \\ 0 \\ 0 \\ s_2 W_1 \end{Vmatrix}$$

and

$$s_1 = \frac{1}{R_0 U \cos\varphi}, \qquad s_2 = \frac{1}{R_0} \tag{2.5}$$

From Equation (2.3) we can, in the usual way, obtain the matrix integral equation

$$x(t) = N(t)\, x(0) + \lambda \int_0^t N(t-\tau_1)\, z(\tau_1)\, d\tau_1 - \lambda \int_0^t N(t-\tau_1)\, y(\tau_1)\, x(\tau_1)\, d\tau_1 \tag{2.6}$$

where $N(t) = \| N_{jk}(t) \|$ $(j, k = 1, \ldots, 4)$ is the matrix weight function for the homogeneous matrix differential equation

$$dx/dt + ax = 0 \tag{2.7}$$

Corresponding to the differential equation (2.7), the characteristic Equation

$$\rho^4 + (\nu^2 + \mu^2 + 2\Omega^2)\rho^2 + \nu^2\mu^2 - (\nu^2 + \mu^2)\Omega^2 + \Omega^4 = 0$$

has two pairs of imaginary roots

$$\rho_1, \rho_2 = \pm i\omega_1, \qquad \rho_3, \rho_4 = \pm i\omega_2$$

The function $N_{jk}(t)$ in Equation (2.6) can be represented as [2]

$$\begin{aligned} N_{jk}(t) &= -\frac{1}{e_1} T_{jk}^{(1)} \cos\omega_1 t + \frac{1}{e_2} T_{jk}^{(2)} \cos\omega_2 t \qquad \text{for } j + k = 2m \\ N_{jk}(t) &= \frac{1}{e_1} T_{jk}^{(1)} \sin\omega_1 t - \frac{1}{e_2} T_{jk}^{(2)} \sin\omega_2 t \qquad \text{for } j + k = 2m + 1 \end{aligned} \tag{2.8}$$

where m is a positive integral number.

Here $e_s = \omega_s(\omega_2^2 - \omega_1^2)$ $(s = 1, 2)$, and the functions $T_{jk}^{(s)}$ $(s = 1, 2)$ are

$$\begin{aligned}
&T_{11}^{(s)} = \omega_s^3 - (\mu^2 + \Omega^2)\omega_s, && T_{12}^{(s)} = -\nu^2\omega_s^2 - \Omega^2\nu^2 + \mu^2\nu^2 \\
&T_{21}^{(s)} = \omega_s^2 + \Omega^2\mu^2/\nu^2 - \mu^2, && T_{22}^{(s)} = \omega_s^3 - (\mu^2 + \Omega^2)\omega_s \\
&T_{31}^{(s)} = -(\Omega + \Omega\mu^2/\nu^2)\omega_s, && T_{32}^{(s)} = \Omega\omega_s^2 + \Omega\mu^2 - \Omega^3 \\
&T_{41}^{(s)} = -\Omega\omega_s^2 - \Omega\nu^2 + \Omega^3, && T_{42}^{(s)} = -2\Omega\nu^2\omega_s \\
&T_{13}^{(s)} = -2\Omega\nu^2\omega_s, && T_{14}^{(s)} = \Omega\omega_s^2 + \Omega\mu^2 - \Omega^3 \\
&T_{23}^{(s)} = -\Omega\omega_s^2 - \Omega\nu^2 + \Omega^3, && T_{24}^{(s)} = -(\Omega + \Omega\mu^2/\nu^2)\omega_s \\
&T_{33}^{(s)} = \omega_s^3 - (\nu^2 + \Omega^2)\omega_s, && T_{34}^{(s)} = -\omega_s^2\mu^2/\nu^2 - \Omega^2 + \mu^2 \\
&T_{43}^{(s)} = \nu^2\omega_s^2 + \Omega^2\nu^2 - \nu^4, && T_{44}^{(s)} = \omega_s^3 - (\nu^2 + \Omega^2)\omega_s
\end{aligned} \tag{2.9}$$

The solution of the matrix integral equation (2.6) can be found by successive substitutions [3].

Denoting

$$F(t) = N(t)\, x(0) + \lambda \int_0^t N(t-\tau_1)\, z(\tau_1)\, d\tau_1 \tag{2.10}$$

we can bring Equation (2.6) to the form

$$x(t) = F(t) - \lambda \int_0^t N(t-\tau_1)\, y(\tau_1)\, x(\tau_1)\, d\tau_1 \tag{2.11}$$

From (2.11) it follows that

$$x(\tau_1) = F(\tau_1) - F \int_0^{\tau_1} N(\tau_1 - \tau_2)\, y(\tau_2)\, x(\tau_2)\, d\tau_2 \tag{2.12}$$

Substituting into (2.11) the expression (2.12) of $x(\tau_1)$, we obtain

$$x(t) = F(t) - \lambda \int_0^t N(t-\tau_1)\, y(\tau_1) \left[F(\tau_1) - \lambda \int_0^{\tau_1} N(\tau_1 - \tau_2)\, y(\tau_2)\, x(\tau_2)\, d\tau_2\right] d\tau_1 \quad (2.13)$$

Substituting $x(\tau_2)$ into (2.13) with an expression analogous to (2.12), and repeating the described process, we can represent the solution of the integral equation (2.6) in the form of an absolutely and uniformly converging series [3], which, in the present problem, will be a power series with a small parameter λ

$$x(t) = F(t) + \sum_{n=1}^{\infty} V_n(t) \quad (2.14)$$

where

$$V_n(t) = (-1)^n \lambda^n \int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_{n-1}} N(t-\tau_1)\, N(\tau_1 - \tau_2) \ldots$$

$$\ldots N(\tau_{n-1} - \tau_n)\, y(\tau_1) \ldots y(\tau_n)\, F(\tau_n)\, d\tau_1 \ldots d\tau_n \quad (2.15)$$

Restricting ourselves to terms not higher than of second order of small magnitude, we can represent the solution (2.14) in the form

$$x(t) = \left(N(t) - \lambda \int_0^t N(t-\tau_1)\, y(\tau_1) N(\tau_1) d\tau_1 + \lambda^2 \int_0^t \int_0^{\tau_1} \Psi(t, \tau_1, \tau_2)\, d\tau_2\, d\tau_1\right) x(0) +$$

$$+ \lambda \int_0^t N(t-\tau_1)\, z(\tau_1)\, d\tau_1 - \lambda^2 \int_0^t \int_0^{\tau_1} Q(t, \tau_1, \tau_2)\, d\tau_2\, d\tau_1 \quad (2.16)$$

where

$$\Psi(t, \tau_1, \tau_2) = N(t-\tau_1)\, y(\tau_1)\, N(\tau_1 - \tau_2)\, y(\tau_2)\, N(\tau_2)$$

$$Q(t, \tau_1, \tau_2) = N(t-\tau_1)\, y(\tau_1)\, N(\tau_1 - \tau_2)\, z(\tau_2) \quad (2.17)$$

With zero initial conditions $x_k(0) = 0$ $(k = 1, \ldots, 4)$ the elements of the matrix (2.16) will have the form

$$x_j(t) = \lambda s_2 \int_0^t [N_{j1}(t-\tau_1)\, W_2(\tau_1) + N_{j4}(t-\tau_1)\, W_1(\tau_1)]\, d\tau_1 -$$

$$- \lambda^2 \int_0^t \int_0^{\tau_1} Q_j(t, \tau_1, \tau_2)\, d\tau_2\, d\tau_1 \quad (j = 1, \ldots, 4) \quad (2.18)$$

where

$$Q_j(t, \tau_1, \tau_2) = s_1 s_2 N_{j1}(t-\tau_1)\, N_{14}(\tau_1 - \tau_2)\, W_1(\tau_1)\, W_1(\tau_2) -$$

$$- s_1 s_2 [N_{j4}(t-\tau_1)\, N_{11}(\tau_1 - \tau_2) + N_{j2}(t-\tau_1)\, N_{31}(\tau_1 - \tau_2)]\, W_2(\tau_1)\, W_2(\tau_2) +$$

$$+ s_1 s_2 N_{j1}(t-\tau_1)\, N_{11}(\tau_1 - \tau_2)\, W_1(\tau_1)\, W_2(\tau_2) -$$

$$- s_1 s_2 [N_{j4}(t-\tau_1)\, N_{14}(\tau_1 - \tau_2) + N_{j2}(t-\tau_1)\, N_{34}(\tau_1 - \tau_2)]\, W_2(\tau_1)\, W_1(\tau_2) -$$

$$- s_2^2 [N_{j1}(t-\tau_1)\, N_{24}(\tau_1 - \tau_2) - N_{j4}(t-\tau_1)\, N_{34}(\tau_1 - \tau_2)]\, W_3(\tau_1)\, W_1(\tau_2) -$$

$$- s_2^2 [N_{j1}(t-\tau_1)\, N_{21}(\tau_1 - \tau_2) - N_{j4}(t-\tau_1)\, N_{31}(\tau_1 - \tau_2)]\, W_3(\tau_1)\, W_2(\tau_2)$$

$$(j = 1, \ldots, 4) \quad (2.19)$$

Since the mathematical expectations of the random processes $W_i(t)$ $(i = 1, 2, 3)$ are equal to zero, we can represent the mathematical expectations of the random processes $x_j(t)$, which determine the position of the gyrocompass, in the following form:

$$M[x_j(t)] = -\lambda^2 \int_0^t \int_0^{\tau_1} M[Q_j(t, \tau_1, \tau_2)]\, d\tau_2\, d\tau_1 \qquad (j = 1, \ldots, 4) \quad (2.20)$$

$$M\left[Q_j(t,\tau_1,\tau_2)\right] = s_1 s_2 N_{j1}(t-\tau_1) N_{14}(\tau_1-\tau_2) K_{11}(\tau_1-\tau_2) -$$
$$- s_1 s_2 \left[N_{j4}(t-\tau_1) N_{11}(\tau_1-\tau_2) + N_{j2}(t-\tau_1) N_{31}(\tau_1-\tau_2)\right] K_{22}(\tau_1-\tau_2) +$$
$$+ s_1 s_2 N_{j1}(t-\tau_1) N_{11}(\tau_1-\tau_2) K_{12}(\tau_1-\tau_2) - s_1 s_2 \left[N_{j4}(t-\tau_1) N_{14}(\tau_1-\tau_2) +\right.$$
$$\left. + N_{j2}(t-\tau_1) N_{34}(\tau_1-\tau_2)\right] K_{12}(\tau_2-\tau_1) -$$
$$- s_2^2 \left[N_{j1}(t-\tau_1) N_{24}(\tau_1-\tau_2) - N_{j4}(t-\tau_1) N_{34}(\tau_1-\tau_2)\right] K_{13}(\tau_2-\tau_1) -$$
$$- s_2^2 \left[N_{j1}(t-\tau_1) N_{21}(\tau_1-\tau_2) - N_{j4}(t-\tau_1) N_{31}(\tau_1-\tau_2)\right] K_{23}(\tau_2-\tau_1)$$
$$(j = 1, \ldots, 4) \tag{2.21}$$

Here $K_{ij}(\tau_1-\tau_2)$ are the correlation functions of the random processes $W_k(t)$ $(k = 1, 2, 3)$

$$K_{ij}(\tau_1-\tau_2) = M\left[W_i(\tau_1) W_j(\tau_2)\right] \qquad (i, j = 1, 2, 3) \tag{2.22}$$

We assume that processes $W_k(t)$ are stationary and stationary connected. Therefore the correlation functions depend on the difference of the arguments.

The variances of the random processes $x_j(t)$, according to (2.18), will, with an accuracy up to terms of second order of smallness, have the following form:

$$D_j(t) = \lambda^2 s_2^2 \int_0^t \int_0^t \{N_{j4}(t-\tau_1) N_{j4}(t-\tau_2) K_{11}(\tau_1-\tau_2) + N_{j1}(t-\tau_1) N_{j1}(t-\tau_2) \times$$
$$\times K_{22}(\tau_1-\tau_2) + 2N_{j4}(t-\tau_1) N_{j1}(t-\tau_2) K_{12}(\tau_1-\tau_2)\}\, d\tau_2\, d\tau_1 \qquad (j = 1, \ldots, 4) \tag{2.23}$$

3. As an example, we shall investigate the motion of the gyrocompass when $W_j(t)$ $(j = 1, 2, 3)$ are stationary processes of the white noise type, with zero mathematical expectations

$$M\left[W_j(t)\right] = 0 \qquad (j = 1, 2, 3) \tag{3.1}$$

and correlation functions of the form

$$\begin{aligned} K_{11}(t-\tau) &= G_1 \delta(t-\tau), & K_{12}(t-\tau) &= 0 \\ K_{22}(t-\tau) &= G_2 \delta(t-\tau), & K_{13}(t-\tau) &= L\delta(t-\tau) \\ K_{33}(t-\tau) &= G_3 \delta(t-\tau), & K_{23}(t-\tau) &= 0 \end{aligned} \tag{3.2}$$

which can take place with vibrations of the instrument base. Here $\delta(t-\tau)$ is a delta function. Note, that the problem of the gyrocompass motion due to heaving, determined, for instance, with the relations published by Sveshnikov [4], would require the calculation in the random processes $W_j(t)$ of nonlinear terms of the second order relative to the angles of heaving of the ship and their derivatives. These nonlinear terms, as demonstrated in the paper [4], would by themselves influence the magnitudes of the mathematical expectations of the angles, determining the position of the gyrocompass.

We shall calculate the mathematical expectations $M[x_j(t)]$ according to (2.20). Since

$$\int_0^{\tau_1} N_{jk}(\tau_1-\tau_2)\, \delta(\tau_1-\tau_2)\, d\tau_2 = \frac{1}{2} N_{jk}(0)$$

and the matrix weight function $N(t)$ satisfies the condition $N(0) = E$, where E is the unit matrix, then, using (3.2), we can write (2.20) as

$$M\left[x_j(t)\right] = \frac{1}{2}\lambda^2 s_1 s_2 G_2 \int_0^t N_{j4}(t-\tau_1)\, d\tau_1 \qquad (j = 1, \ldots, 4) \tag{3.3}$$

Substituting λ, s_1 and s_2 in (3.3) by their expressions, we get

$$M\left[x_j(t)\right] = \frac{G_2}{2R^2 U \cos\varphi} \int_0^t N_{j4}(\xi)\, d\xi \tag{3.4}$$

Integrating (3.4), we can present the mathematical expectations of the angles, determining the position of the gyrocompass, as follows:

$$M[x_j(t)] = \frac{G_2}{2R^2U\cos\varphi}\Lambda_{j4}(t) \qquad (j = 1, \ldots, 4) \tag{3.5}$$

$$\Lambda_{j4}(t) = \begin{cases} T_{j4}^{(1)}\dfrac{1}{\omega_1 e_1}(1 - \cos\omega_1 t) - T_{j4}^{(2)}\dfrac{1}{\omega_2 e_2}(1 - \cos\omega_2 t) & (j = 1, 3) \\ -T_{j4}^{(1)}\dfrac{1}{\omega_1 e_1}\sin\omega_1 t + T_{j4}^{(2)}\dfrac{1}{\omega_2 e_2}\sin\omega_2 t & (j = 2, 4) \end{cases} \tag{3.6}$$

The variances of the random processes $x_j(t)$, according to (2.23) and (3.2) will be

$$D_j(t) = a_j t + b_j\sin(\omega_1 - \omega_2)t + c_j\sin 2\omega_1 t + h_j\sin 2\omega_2 t + l_j\sin(\omega_1 + \omega_2)t \qquad (j = 1, \ldots, 4) \tag{3.7}$$

where

$$a_j = \frac{1}{2R^2}[G_1(m_{j4}^2 + n_{j4}^2) + G_2(m_{j1}^2 + n_{j1}^2)], \qquad b_j = -\frac{1}{(\omega_1 - \omega_2)R^2}(G_1 m_{j4}n_{j4} + G_2 m_{j1}n_{j1})$$

$$c_j = \frac{(-1)^j}{4\omega_1 R^2}(G_1 m_{j4}^2 - G_2 m_{j1}^2), \qquad h_j = \frac{(-1)^j}{4\omega_2 R^2}(G_1 n_{j4}^2 - G_2 n_{j1}^2) \tag{3.8}$$

$$l_j = \frac{(-1)^j}{(\omega_1 + \omega_2)R^2}(-G_1 m_{j4}n_{j4} + G_2 m_{j1}n_{j1}), \qquad m_{jk} = \frac{1}{e_1}T_{jk}^{(1)}, \qquad n_{jk} = \frac{1}{e_2}T_{jk}^{(2)} \tag{3.9}$$

From (3.5) and (3.7) it can be seen, that even in the absence of initial displacements i.e. with $x_k(0) = 0$ ($k = 1, \ldots, 4$), under conditions of heaving and ship vibrations, a displacement of the average position will occur in in an undamped gyrocompass; the mathematical expectations $M[x_j(t)]$ have a constant component. The gyrocompass performs undamped quasiperiodic vibrations about this displaced position.

The variances of the angles, determining the position of the gyrocompass, in addition to periodic components, also contain a component linearly increasing with time. After a sufficiently long time, the variance of the undamped gyrocompass can reach a significant value.

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